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NOTES ON THE GEOMETRICAL REPRESENTATION OF THE ROOTS OF EQUATIONS.

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The importance, from the historical standpoint, of the method of representing the roots of equations by means of intersecting conics, suggests that this species of graphical representation might deserve a more prominent place in teaching. Constructions of this kind were known to the Greek geometers, an account of whose discoveries may be found in Zeuthen's *Die Lehre von den Kegelschnitten im Altertum*, Chap. 11, pp. 240-243. An account of the work in the same direction by Descartes, Van Schooten and others is given in M. Cantor's *Geschichte der Mathematik*, Vol. 2, Chap. LXXVII, pp. 736-737, and a short reference to the methods in Klein's ** Vortraege über Ausgewahlte Fragen der Elementargeometrie.*

In this paper constructions for the cubic, biquadratic, and quintic are obtained in what is believed to be their most practicable form. These results are then employed to derive elementary analytical properties of the equations directly from the geometrical properties of the constructions.

§1. THE CUBIC.

The roots of the general cubic

$$x^3+px+q=0 \tag{1}$$

may be obtained by eliminating z from the simultaneous pair

$$x^2+qz+p=0,$$
$$x+qz^2+pz=0,$$

x, z being independent variables. Also the same trio of x values will be obtained from this pair altered by the transformation

$$z=\frac{-y}{q}-\frac{p}{q}.$$

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*See Translation by Beman and Smith.

Hence the roots of the cubic (1) are the abscissas of the intersections (apart from the origin) of the parabola and circle

$$(2) \qquad \begin{aligned} P &: y-x^2=0, \\ S &: x^2+y^2+qx+(p-1)y=0. \end{aligned}$$

The parabola P is the same for all cubics, and will be called the primary curve. The circle S always passes through the origin but is different for different cubics. Let us call it the secondary curve. Since the sum of the three abscissas in question is always zero, to each circle through the origin will correspond a cubic equation in form similar to (1) and inversely, so that the totality (∞^2) of all cubics is represented in the plane uniquely by the totality of all circles (∞^2) which pass through the origin O. The secondary S has its center at the rational point

$$\left(-\frac{q}{2}, \frac{1-p}{2}\right)$$

and is always real. Hence, *to construct the roots of the cubic, take* $\left(-\frac{q}{2}, \frac{1-p}{2}\right)$ *as a center C, and with* $\overline{CO}$ *as a radius describe a circle S. The perpendiculars from the intersections of this circle and P, upon the axis of P, are the roots of the cubic*

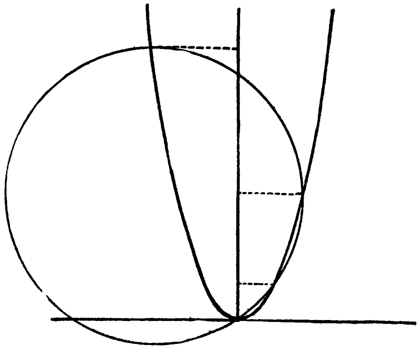
$$x^3+px+q=0.$$

Example: Construct the roots of the equation

$$x^3-7x+6=0.$$

Here we have

$$P: y-x^2=0; \quad S: x^2+y^2+6x-8y=0.$$



I. Scale, $\frac{3}{16}$ in.=unity.

From the figure [1], $r_1=-3; r_2=2; r_3=1$.

Equal, Real, and Imaginary Roots.

If there exists a relation connecting p, q

$$(3) \qquad f(p, q)=0$$

the pencil (∞') of circles S will have an envelope. When S is tangent to P, two roots of the cubic are equal. Hence if the envelope corresponding to the given relation (3) contains P as a factor, $f(p, q)$ is the condition for equal roots.

Now f may be determined to fulfill these conditions. For if we consider as known the theorem that the discriminant of an algebraical equation of degree n is an homogeneous function of its n coefficients, of degree $2(n-1)$, we may write the discriminant for $n=3$ in the following form, with undetermined coefficients α:

$$\Delta : \alpha_1+\frac{\alpha_2}{3}p+\alpha_3 q+\frac{\alpha_4}{27}p^3+\frac{\alpha_5}{4}p^2+\frac{\alpha_6}{4}p^2q+\alpha_7 q^3+\alpha_8 q^2+\frac{\alpha_9}{3}q^2p+\alpha_{10}pq.$$

The elimination of p, q, $\partial q/\partial p$ between the four equations

$$S=0,\ \Delta=0,\ \frac{\partial S}{\partial p}=0,\ \frac{\partial \Delta}{\partial p}=0,$$

and equating to zero (identically) the result of substituting $y=x^2$ in the result gives

$$\alpha_1=\alpha_2=\alpha_3=\alpha_5=\alpha_6=\alpha_7=\alpha_9=\alpha_{10}=0,\ \alpha_4=1,\ \alpha_8=\tfrac{1}{4}.$$

Hence the condition for equal roots is

$$\frac{q^2}{4}+\frac{p^3}{27}=0.$$

Reality of Roots in Cardan's Irreducible Case.

Let $$\frac{q^2}{4}+\frac{p^3}{27}=p^2\delta,\ (\delta\doteq 0).$$

Then $$q=\pm\frac{2}{\sqrt{27}}[(-p)^{\frac{3}{2}}+\tfrac{27}{2}(-p)^{\frac{1}{2}}\delta]$$

and $$S : x^2+y^2\pm\frac{2}{\sqrt{27}}[(-p)^{\frac{3}{2}}+\tfrac{27}{2}(-p)^{\frac{1}{2}}\delta]x+(p-1)y=0,$$

$$\frac{\partial S}{\partial p} : \pm\frac{2}{\sqrt{27}}[(-\tfrac{3}{2})(-p)^{\frac{1}{2}}-\tfrac{27}{4}(-p)^{-\frac{1}{2}}\delta]x+y=0,$$

and the envelope is

$$x^4+x^2y^2-y^3-x^2y+9x^2y\delta=0.$$

That is

$$y=x^2+\frac{9x^2y}{x^2+y^2}\delta.$$

Hence the ordinate of this envelope is greater than the corresponding ordinate of the primary when δ is positive, and less when δ is negative. Thus the envelope of S is enclosed entirely within the primary, or is entirely enclosed by the primary according as δ is positive or negative. Hence when δ is positive,

$$\frac{q^2}{4}+\frac{p^3}{27}>0,$$

two roots of (1) are imaginary. But when δ is negative,

$$\frac{q^2}{4}+\frac{p^3}{27}<0 \text{ (Cardan's irreducible case),}$$

S intersects P in three real points, so all the roots are real.

§ 2. THE QUARTIC.

The roots of the general quartic

$$x^4+px^2+qx+r=0 \tag{4}$$

are the abscissas of the intersections of

$$P : xy-1=0,$$
$$S : x^2+ry^2+qy+p=0.$$

The primary (the same for all quartics) is the equilateral hyperbola P. The secondary is a three parameter central conic. Another convenient form is

$$P' : xy-\sqrt{(\epsilon r)}=0,$$
$$S' : x^2+\epsilon y^2+\frac{q}{\sqrt{\epsilon r}}y+p=0,$$

where $\epsilon=1, -1$ as r is positive or negative. If r is positive S' is a circle with center at $\left(0, \frac{-q}{2\sqrt{r}}\right)$. The center of the secondary conic is always on the y axis. The geometrical construction of the roots is now obvious.

Example: Construct the roots of the quartic

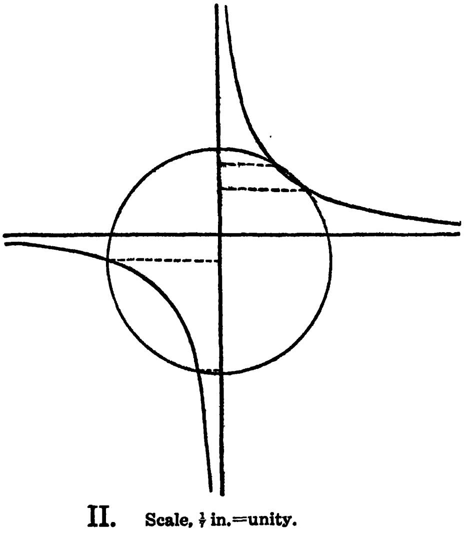
$$x^4-15x^2+10x+24=0.$$

We have

$$P' : xy-2\sqrt{6}=0;$$
$$S' : x^2+y^2+\tfrac{5}{6}\sqrt{6}\,y-15=0.$$

Center $(0, -\frac{5}{12}\sqrt{6})$; radius$=\frac{1}{2}\sqrt{\frac{385}{6}}$. From the figure [II],

$$r_1=-1,\ r_2=-4,\ r_3=2,\ r_4=3.$$



II. Scale, $\frac{1}{7}$ in.=unity.

Harmonic and Equi-Anharmonic Ranges of Roots.

Let us find the envelope of S corresponding to the relation

$$g_2 \,:\, \tfrac{1}{12}(p^2+12r)=0,$$

regarding q as invariable. We have from this equation

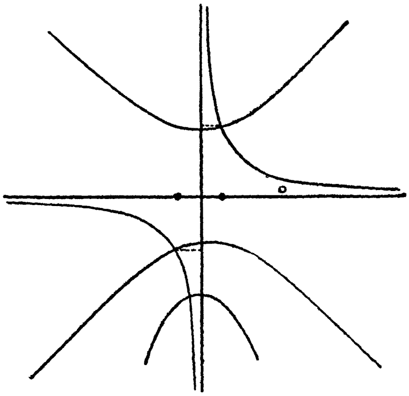
$$S \,:\, x^2-\frac{p^2}{12}y^2+qy+p=0,$$

$$\frac{\partial S}{\partial p} \,:\, -\tfrac{1}{6}y^2p+1=0.$$

Hence $p=\dfrac{6}{y^2}$, and the envelope is

$$(5) \qquad x^2y^2+qy^3+3=0.$$

We may state the result as follows: *When the secondary conic is tangent to the quartic* (5) *the roots of the biquadratic* (4) *form an equi-anharmonic range.* (See example, Fig. III.)



$x^4+6x^2+x-3=0$

III. Scale, $\frac{1}{7}$ in.=unity.

Next consider p as invariable and let

$$g_3 \,:\, \tfrac{1}{6}(pr-\tfrac{3}{8}q^2-\frac{p^3}{36})=0.$$

Then

$$r=\tfrac{3}{8}\frac{q^2}{p}+\frac{p^2}{36},$$

and

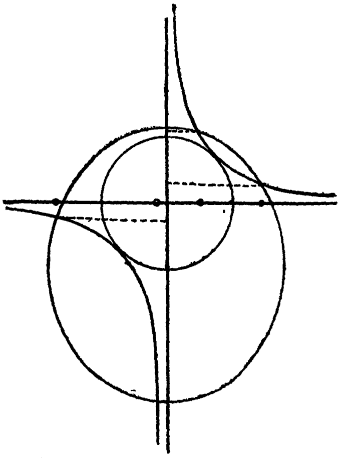
$$S \,:\, x^2+(\tfrac{3}{8}\frac{q^2}{p}+\frac{p^2}{36})y^2+qy+p=0,$$

$$\frac{\partial S}{\partial q} \,:\, \tfrac{3}{4}\frac{q}{p}y^2+y=0;\; q=-\tfrac{4}{3}\frac{p}{y}.$$

And the envelope is the ellipse

$$\frac{x^2}{\frac{-p}{3}}+\frac{y^2}{\frac{-12}{p}}=1.$$

Thus, *when the secondary conic is tangent to this ellipse the roots of the corresponding biquadratic equation form an harmonic range.* (See example, Fig. IV.)



$4x^4-24x^2+8x+3=0$

IV. Scale, $\frac{7}{32}$ in.=unity.

§ 3. THE QUINTIC.

The roots of the general quintic

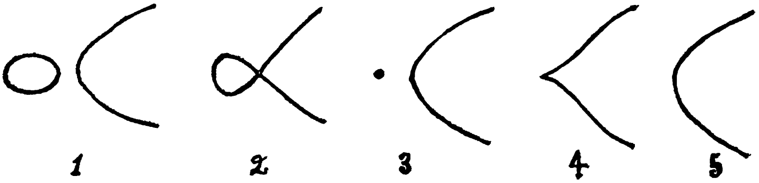
$$x^5+px^3+qx^2+rx+s=0 \tag{6}$$

are the abscissas of the five finite points of intersection of the curves

$$P: xy-1=0,$$
$$S: x^2+sy^3+ry^2+qy+p=0.$$

The second is a Newton divergent parabola.

There are five non-projective classes of the latter curves, as follows:



1. $sx^2=(y-\alpha)(y-\beta)(y-\gamma)$; all real factors, and $\alpha\neq\beta\neq\gamma$.
2. $sx^2=(y-\alpha)(y-\beta)^2$, $\alpha<\beta$; all real factors.
3. $sx^2=(y-\beta)^2(y-\alpha)$, $\beta>\alpha$; all real factors.
4. $sx^2=(y-\beta)^3$; three coincident roots.
5. $sx^2=(y^2+ay+b)(x-c)$; two imaginary roots.

Example: Construct the roots of the quintic equation

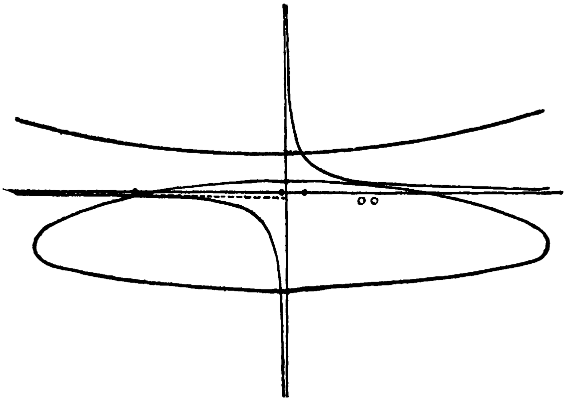
$$x^5-21x^3+50x^2-12x-8=0.$$

The secondary curve is

$$x^2-8y^3-12y^2+50y-21=0.$$

That is

$$\frac{x^2}{8}=(y-\tfrac{1}{2})(y-\tfrac{3}{2})(y+\tfrac{7}{2}).$$



VI. Scale, $\frac{3}{16}$ in.=unity.

Hence the secondary cubic is of type 1. From figure VI we have $r_1=.64+$, r_2 and r_3 imaginary, $r_4=-.3+$, $r_5=-5.4+$.

It is evident that the only classes of the Newtonian parabolas that can intersect the primary hyperbola in five real points are 1 and 2. Hence a necessary condition that all of the roots of a quintic (6) be real is that it be possible to resolve

$$sy^3+ry^2+qy+p=0$$

into real factors. That is, that the following relation hold:

$$(9ps-qr)^2-4(3qs-r^2)(3pr-q^2)\leqq 0. \tag{7}$$

This condition, though necessary, is not sufficient.

§ 4. A Reducible Case of the Sextic.

The sextic

$$x^6+px^4+qx^3+rx^2+sx+t=0 \tag{8}$$

is equivalent to the simultaneous equations

$$xz-1=0;\ x^3+px+tz^3+sz^2+rz+q=0.$$

In these z may be transformed linearly, taking

$$z=t^{-\frac{1}{3}}y+h$$

where h is a root of the cubic

$$th^3+sh^2+rh+q=0. \tag{9}$$

This gives for the primary and secondary curves

$$\begin{aligned} &P:\ xy+t^{\frac{1}{3}}hx-t^{\frac{1}{3}}=0,\\ &S:\ x^3+px+y^3+uy^2+vy=0,\\ \text{where}\quad &u=t^{-\frac{2}{3}}(3ht+s);\ v=t^{-\frac{1}{3}}(3th^2+2sh+r). \end{aligned}$$

The secondary cubic passes through the origin and has the asymptote

$$a_1:\ y=-x-\tfrac{1}{3}u.$$

Now S may degenerate into a straight line and a conic. In fact S degenerates if, and only if, one of the two following conditions is fulfilled:

$$\begin{aligned} &d_2:\ v=p;\ u=0,\\ &d_2:\ u=3\sqrt{-p};\ v=-2p\ (p \text{ necessarily negative}). \end{aligned}$$

If $v\neq p$ and $u=0$, the curve S has the origin for a center of symmetry. All these cubics cross the fixed asymptote $y=-x$ at the origin. The origin is a point of inflexion, the inflexional tangent being

$$T:\ y=-\frac{p}{v}x.$$

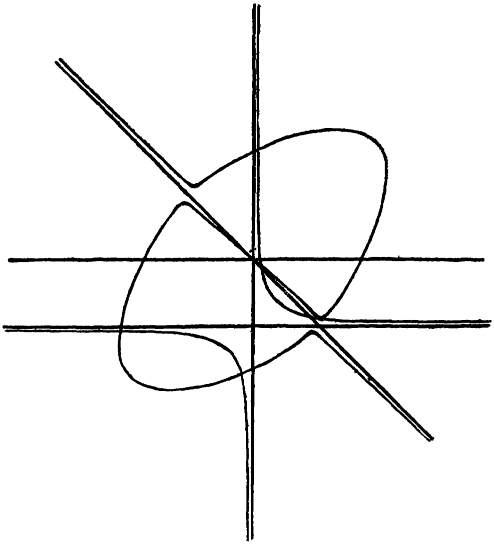
Thus when $u=0$, *i. e.*, when

$$2s^3-9rst+27qt^2=0,$$

the variation of the secondary, corresponding to changes in the coefficients of the sextic (8), is characterized by the rotation of T around the origin, and when T coincides with the asymptote, S degenerates into straight line and ellipse

$$S:L.C\equiv(x+y)(x^2-xy+y^2+v)=0,$$

i. e., into the asymptote itself and the ellipse $C=0$, origin at center and axis coinciding with the asymptote.



VII. Scale, ⅛ in.=unity.

Figure [VII] shows the curves S and P when the sextic (8) is approaching

$$(10)\quad x^6-25.9x^4+50.6x^3-2x^2-4x+\tfrac{1}{3}=0.$$

When condition d_2 holds we have

$$S:\ a_1C_1\equiv(x+y+\sqrt{-p})[x^2-xy+y^2+2\sqrt{(-p)}y-\sqrt{(-p)}x]$$

Thus S degenerates into the asymptote, and the ellipse C_1 through the origin, center at the point where the asymptote crosses the y axis, axis coinciding with the asymptote. The sextic equations corresponding to d_1, d_2 may be solved by solving the cubic (9), extracting the roots $\sqrt[3]{t}$, $\sqrt{-p}$, and solving a quadratic and a biquadratic. In fact the sextics corresponding to d_1, d_2 are respectively,

$$(11)\quad (x^2-t^{\frac{1}{3}}hx+t^{\frac{1}{3}})[x^4+t^{\frac{1}{3}}hx^3+(t^{\frac{2}{3}}h^2-t^{\frac{1}{3}}+p)x^2-2t^{\frac{2}{3}}hx+t^{\frac{2}{3}}]=0.$$

$$(12)\quad [x^2+(\sqrt{(-p)}-t^{\frac{1}{3}}h)x+t^{\frac{1}{3}}][x^4+(t^{\frac{1}{3}}h-\sqrt{-p})x^3$$
$$+(t^{\frac{2}{3}}h^2-2\sqrt{(-p)}t^{\frac{1}{3}}h-t^{\frac{1}{3}})x^2+(2\sqrt{(-p)}t^{\frac{1}{3}}-2t^{\frac{2}{3}}h)x+t^{\frac{2}{3}}]=0.$$